LXII. *A Letter to the Right Honourable* George *Earl of* Macclesfield, *P. R. S. concerning the Value of an Annuity for Life, and the Probability of Survivorſhip. By Mr.* James Dodſon.

My Lord,

Read Jan. 17, 1754.

YOUR Lordſhip's approbation of my eſſay, concerning the method of forming the ſeries for making logarithms, and that for rectifying the circle, without the aſſiſtance of fluxions, has emboldened me to lay before you the following inveſtigations, concerning the value of an annuity for life (ſecured by land), and the probability of ſurvivorſhip, between two perſons of given ages; neither of which have (as I believe) been before attempted to be found, otherwiſe than by fluxions; at leaſt, no ſuch attempt has appeared in public.

I am ſenſible that I ought to apologize for reciting the principles, upon which calculations of this kind are commonly founded, becauſe I know that your Lordſhip is perfectly well acquainted with them; but I plead in excuſe thereof, that I found a great difficulty would attend the rendring my argument intelligible and concluſive, if they were omitted. I am, with the greateſt deference, my Lord,

Your Lordſhip's moſt obliged,

and moſt humble ſervant,

Bell-Dock, Wapping,
Oct. 18, 1753.

J. Dodſon.

The Royal Society is collaborating with JSTOR to digitize, preserve, and extend access to *Philosophical Transactions (1683-1775).* www.jstor.org

The writers upon the ſubject of annuities on lives have very juſtly diſtinguiſhed them into two kinds: In the firſt, the annuitant is entituled to receive a payment, if he be alive on the day on which it becomes due; but if he dies on the preceding day, or ſooner, his heirs have no claim to any part of the payment, ſo to have became due; but in the ſecond, if the annuitant dies at any intermediate time, between the days of payment, his heirs are to receive a part of the annuity, proportional to the time elapſed, between the preceding day of payment and the annuitant's deceaſe.

This latter kind of annuities have been diſtinguiſhed from the former, by the words, *ſecured by a grant of lands*; becauſe, where lands are leaſed for lives, the conditions are generally ſuch as are above deſcribed.

The values of the firſt kind of annuities have been inveſtigated upon principles purely arithmetical; but, in order to perform the latter, fluxions have been uſed (as I humbly conceive) without any neceſſity: But as the inveſtigation of the former may be uſefully made a part of the latter, I ſhall firſt recite the method of performing that, and then proceed to attempt the other, upon the ſame principles.

If, with the ſagacious mathematician Mr. Abraham De Moivre, we ſuppoſe the decrements of life to be equal (*viz.* that out of a number of perſons, alive at a given age, equal to the number of years that a perſon of that age hath a poſſibility of living, there will die one in each year, till they are all extinct); then, out of a number of chances equal to that number of perſons, which may, for inſtance, be be 36, all but one are favourable, in the firſt year, to any individual; and, conſequently, it is 35 to 1, that he receives one payment of the annuity, by living till it becomes due; that is, the probability of his receiving it, is $\frac{35}{36}$, and that of his not receiving it, $\frac{1}{36}$.

Again; ſince, by ſuppoſition, there dies but one perſon in the firſt year, and one in the ſecond; there are but two chances, in the 36, againſt his receiving the ſecond payment, by living till it becomes due; and, conſequently, $\frac{34}{36}$ will be the probability of his receiving that alſo; the probability of his dying in that year being $\frac{1}{36}$, as before.

In like manner it may be proved, that the probability of his receiving the third, fourth, fifth, *&c.* payment, will be $\frac{33}{36}$, $\frac{32}{36}$, $\frac{31}{36}$, *&c.* and therefore, if the annual payments were each 1 *l.* and if the intereſt of money was not to be conſidered, we might conceive theſe ſeveral probabilities, as the preſent worths of the ſeveral payments, and the ſum of them would be the value of an annuity of the firſt kind.

But ſince the intereſt of money neceſſarily enters the proceſs, and ſince the payments become due at the end of the firſt, ſecond, third, *&c.* year; therefore the firſt of theſe payments muſt be diſcounted for one year, the ſecond for two years, the third for three years, *&c.* and the ſum of their preſent worths will be the value of an annuity of the firſt kind, to continue during the life of a perſon, who may poſſibly live 36 years; and this ſum may be found by an eaſy and well-known proceſs (from the common tables of compound intereſt and annuities), which need not be inſerted here.

The annuity ſecured by land muſt neceſſarily be of greater value than the above; becauſe, altho' the annuitant dies before the payment becomes due, yet his heirs are to receive a part thereof; the annuitant, therefore, in this caſe, hath not only the probability $\frac{35}{36}$ of receiving the firſt payment, but he hath alſo an expectation upon part of the probability $\frac{1}{36}$, which, in the firſt caſe, was wholly againſt him. Now it may be eſteemed an equal chance (ſuppoſing him to die in the firſt year) whether that deceaſe happens before the expiration of half that year, or after it; and if it happens before, he is to receive leſs than half the annual payment; but if after it, more.

The annuitant may, therefore, be ſuppoſed to have an equal chance, if he fails of receiving the whole firſt payment, yet of receiving half thereof; and, conſequently, half of the probability, $\frac{1}{36}$, which was before totally againſt him, will (in this caſe) be favourable to him; and his expectation of receiving either the whole, or at leaſt half of the firſt payment, will be $\frac{35}{36} + \frac{1}{72}$.

In like manner, ſince the probability of his dying, in the ſecond year, is, alſo, $\frac{1}{36}$; we may (by arguing in the ſame manner) prove, that one half thereof will (in this caſe) become favourable to him; and, conſequently, that $\frac{34}{36} + \frac{1}{72}$ will be the probability of his receiving the whole, or at leaſt half, of the ſecond payment.

It appears, therefore, that, for every year which he hath the poſſibility of living, he will (in this caſe) have the probability ($\frac{1}{72}$ or) half of $\frac{1}{36}$ in his favour, more than he had in the former caſe; and therefore, if the preſent worths of the conſtant ſum $\frac{1}{72}$ *l.* (ſuppoſed

posed to be due at the end of one, two, three, &c. years) be found, and added to the value of the annuity, according to the former case, the sum will be the value of an annuity, secured by land, to continue during the life of a person who may possibly live 36 years.

Now since the sums, whereby the former annuity is to be increased, consist of the present worths of that fraction of a pound sterling (whose numerator is unity, and denominator twice the number of years that the annuitant can possibly live) supposed to be due at the end of each of one, two, three, &c. years; it will follow, that their amount, or the difference between the values of the two annuities, will be equal to the quotient, found by dividing the value of an annuity of 1 *l.* certain, for as many years as the annuitant can possibly live, by twice that number of years: And, therefore, if to the value of an annuity for life, of the first kind, we add the quotient so found; the sum will be the value of an annuity, of the second kind, for the same life.

When I had thus investigated the value of this annuity, I compared the result with that Mr. De Moivre has deduced from fluxions, which is published in N° 473. of the *Philosophical Transactions*; and found, that they agree to more than a sufficient exactness, for computations of this nature: I have, therefore, annexed that comparison hereto.

The above-mentioned author, and others, have also calculated the probabilities of survivorship, by a fluxional process; which probabilities will flow from what is above premised, by a very easy calculus; in the doing which, the above yearly probabilities of the an-

 nuitant

nuitant's receiving the whole firſt, ſecond, third, *&c.* payment, or at leaſt the halves of them (being, in fact, the probabilities of his living the whole of thoſe years, or at leaſt the halves of them), are called the expectations of life, in thoſe years.

Mr. De Moivre has proved, on another occaſion, that, if there be an expectation, to take place upon the happening of two independent events, the probability of the compound event, will be the product of the two ſingle probabilities: Now, if there be an expectation, depending on one man's ſurviving another, it muſt neceſſarily be compounded of the probabilities of the one's living, and of the other's dying. Now let the perſon, upon whoſe ſurviving the expectation depends, be of ſuch an age, that he may poſſibly live 36 years; and let the perſon, who is to be ſurvived, be of that age, that he may poſſibly live 43 years: Then ſince the ſurvivor's expectation of life will, in the firſt, ſecond, third, *&c.* year, be $\frac{35}{36}+\frac{1}{72}$, $\frac{34}{36}+\frac{1}{72}$, $\frac{33}{36}+\frac{1}{72}$, *&c.* and ſince the probability of the other's dying, in any one year, will be $\frac{1}{43}$; therefore, if the former be ſeverally multiplied by the latter, the products ariſing, *viz.* $\overline{\frac{35}{36}+\frac{1}{72}}\times\frac{1}{43}$, $\overline{\frac{34}{36}+\frac{1}{72}}\times\frac{1}{43}$, $\overline{\frac{33}{36}+\frac{1}{72}}\times\frac{1}{43}$, *&c.* will exhibit the probabilities of the ſurvivorſhip's taking place, in the firſt, ſecond, third, *&c.* years.

For the above expectations contain the probabilities of the ſurvivor's living the whole, or (at leaſt) the halves of the firſt, ſecond, third, *&c.* years; and the conſtant factor, $\frac{1}{43}$, is the probability of the other's dying, ſome time within one of thoſe years; which may, therefore, be either in the firſt, or in the ſecond, half year, of either of them; thro' both of which

periods,

periods, the probability of the ſurvivor's living is above exhibited; and, conſequently, the probabilities of the ſurvivorſhip's taking place, in thoſe times, will be ſeverally repreſented by the above products.

Now the ſum of 36 terms of this ſeries of products, will, upon computation, appear to be $\frac{18}{43}$; which is, therefore, the probability of the ſurvivorſhip required.

But if, inſtead of requiring, as above, the probability of the elder perſon's ſurviving the younger, it were required to find the probability of the younger perſon's ſurviving the elder; then, ſince it is almoſt a certainty, that both of them will not die in the ſame moment of time; we may, by denoting that certainty by unity (agreeable to another principle eſtabliſhed in the doctrine of chances) determine the probability required to be $(1 - \frac{18}{43}$ or$)$ $\frac{25}{43}$: The algebraic inveſtigation of each of theſe caſes is annexed, to which I beg leave to refer, for a farther illuſtration; and ſhall only obſerve, farther, that the above two reſults will, upon compariſon with thoſe given, upon a fluxional proceſs by Mr. De Moivre, in his treatiſe of annuities on lives, appear to coincide with them exactly.

The probability of any order of ſurvivorſhip, that can happen among three perſons, and, conſequently, that of one perſon's ſurviving two others, may, likewiſe, be inveſtigated upon ſimilar principles, without the aſſiſtance of fluxions; but as this problem admits of ſix caſes, and the algebraic proceſs is of a length, too great for the deſigned limits of this eſſay, I beg leave to poſtpone it.

A com-

A comparison of the result of the beforegoing investigation of the value of an annuity for life (secured by land), with the value thereof, given by Mr. De Moivre, *in p.* 67. *of* N° 473. *of the* Philosophical Transactions.

If the annual payment, or rent, be supposed to be 1 *l.*; if the rate of interest (that is, the amount of 1 *l.* and its interest for one year) be denoted by r; the number of years which the annuitant may possibly live (called by Mr. De Moivre the complement of life) by n; and the value of an annuity of 1 *l. per annum*, certain for n years, by P: Then, by *Prob.* I. of Mr. De Moivre's treatise of annuities on lives, the value of an annuity (not secured by land) for the life of a person, whose complement is n, will be $\frac{1 - \frac{r}{n}P}{r - 1}$; to which if $\frac{P}{2n}$ be added, the sum $\frac{1 - \frac{r}{n}P}{r - 1} + \frac{P}{2n}$ will, according to the result of the above investigation, be the value of an annuity (secured by land) for the life of the same person: And if the hyperbolic logarithm of r be denoted by α; then, by the above-quoted *Philosophical Transaction*, $\frac{1}{r - 1} - \frac{P}{\alpha n}$ will be the value of the same annuity, according to Mr. De Moivre: Let us proceed, therefore, to examine, whether

ther these two expressions, *viz.* $\frac{1 - \frac{r}{n} P}{r - 1} + \frac{P}{2n}$ and $\frac{1}{r - 1} - \frac{P}{\alpha n}$ are equal, or nearly so.

It is well known, that the hyperbolic logarithm of any number, r, is equal to the infinite series, $\frac{2}{1} \times \frac{r-1}{r+1} + \frac{2}{3} \times \overline{\frac{r-1}{r+1}}\Big|^3 + \frac{2}{5} \times \overline{\frac{r-1}{r+1}}\Big|^5$ &c. and, in this case (since the interest of money has not, for many years, exceeded 5*l.* *per cent.* and is continually decreasing), r will be always expounded by some of the following numbers, 1,05. 1,04. 1,03. &c. or others intermediate to them; in the greatest of which (*viz.* 1,05) $r - 1$ will be expounded by (,05 or) $\frac{1}{20}$, and $r + 1$ by (2,05 or) $\frac{41}{20}$; whence, $\frac{2}{1} \times \frac{r-1}{r+1}$, the first term of the series will be $\frac{2}{41}$, and $\frac{2}{3} \times \overline{\frac{r-1}{r+1}}\Big|^3$ will be $\left(\frac{2}{3} \times \frac{1}{41^3} = \right) \frac{2}{206763}$, a fraction too small to affect a calculation of this kind; and therefore, $\frac{2}{1} \times \frac{r-1}{r+1}$, the first term of the series, will be nearly equal to α, and may be wrote for it.

Now by writing $\frac{2}{1} \times \frac{r-1}{r+1}$ for α, the value of the expression, $\frac{1}{r-1} - \frac{P}{\alpha n}$, will become $\frac{1}{r - 1} -$

$r + 1$

$\frac{\overline{r+1}\times P}{\overline{r-1}\times 2n}$; and it remains, only, to prove the equality of the expressions, $\frac{1-\frac{\ }{n}P}{r-1}+\frac{P}{2n}$ and $\frac{1}{r-1}-\frac{\overline{r+1}\times P}{\overline{r-1}\times 2n}$, which may be done as follows:

Put $\frac{1-\frac{r}{n}P}{r-1} + \frac{P}{2n} = N.$

Then $1-\frac{r}{n}P+\frac{r-1}{2n}P=\overline{r-1}\times N;$

But, because $\frac{r}{n}=\frac{2r}{2n},$

And $\frac{2r}{2n}-\frac{r-1}{2n}=\frac{r+1}{2n};$

Therefore $1-\frac{r+1}{2n}P=\overline{r-1}\times N,$

Whence $\frac{1}{r-1}-\frac{\overline{r+1}\times P}{\overline{r-1}\times 2n}=N:$

And the equality of the two expreſſions is evident: From whence it appears, that the value of an annuity for life (ſecured by land) might have been found, exactly enough, without the aſſiſtance of fluxions.

The investigation of the probabilities of survivorship between two persons of given ages.

Let the complement of the younger life be denoted by n, and that of the elder by p: Then,

First, If the probability of the elder person's surviving the younger be required, let the symbol p be used instead of the number 36, and it will (from the argument above used) appear, that the expectations of the survivor's life, for the first, second, third, &c. years, will be represented by $\left(\frac{p-1}{p}+\frac{1}{2p}, \frac{p-2}{p}+\frac{1}{2p}, \frac{p-3}{p}+\frac{1}{2p}, \text{\&c. or}\right) \frac{2p-1}{2p}, \frac{2p-3}{2p}, \frac{2p-5}{2p}$, &c. which expectations, being severally multiplied into $\frac{1}{n}$, the yearly probability of the other's dying, will give $\frac{2p-1}{2pn}, \frac{2p-3}{2pn}, \frac{2p-5}{2pn}$, &c. for the probability of the survivorship's taking place, in the first, second, third, &c. years.

Now since, by the hypothesis, the survivor cannot possibly out-live p years; therefore, only, p terms of the above series are to be used; which series, being an arithmetical progression, whose greatest term is $\frac{2p-1}{2pn}$, least term $\frac{1}{2pn}$, and number of terms p; the sum thereof will, by a well-known rule, be $\left(\frac{2p-1}{2pn}+\frac{1}{2pn}\times\frac{p}{2}=\frac{2p}{2np}\times\frac{p}{2}=\right)\frac{p}{2n}$; which expreſſion is exactly the ſame as Mr. De Moivre gives, in the 44th page of the 3d edition of his treatiſe of annuities on lives; which by the demonſtration thereof, given in fol, 116. of the ſame work, appears to have been obtained by a fluxional proceſs.

Secondly, If the probability of the younger perſon's ſurviving the elder be required; then, if $\frac{2n-1}{2n}$, $\frac{2n-3}{2n}$, $\frac{2n-5}{2n}$, &c. the yearly expectations of the continuance of the life of the younger, be ſeverally multiplied by $\frac{1}{p}$, the yearly probability of the elder's dying; the products, $\frac{2n-1}{2np}$, $\frac{2n-3}{2np}$, $\frac{2n-5}{2np}$, &c. will give the probabilities of the ſurvivorſhip's taking place, in the firſt, ſecond, third, &c. years.

And here, as before, the ſurvivorſhip muſt neceſſarily take place, before, or at, the end of p years, the longeſt time that the elder can poſſibly live, by ſuppoſition; and therefore, the ſum of p terms, of the ſeries above given, will be the anſwer.

Now this ſeries is, alſo, an arithmetical progreſſion, whoſe greateſt term is $\frac{2n-1}{2np}$; leaſt, $\frac{2n-2p+1}{2np}$; and number of terms, p; whence $\left(\overline{\frac{2n-1}{2np}+\frac{2n-2p+1}{2np}}\times\frac{p}{2}=\frac{4n-2p}{2np}\times\frac{p}{2}=\frac{4n-2p}{2n}\times\frac{1}{2}=\frac{2n-p}{2n}=\right)1-\frac{p}{2n}$ will be the probability required; to which, if $\frac{p}{2n}$, the

pro-

probability of the elder's ſurviving the younger, be added; the ſum will be unity, as was above obſerved.

LXIII. *A Letter to Mr.* Peter Collinſon, *F. R. S. concerning the Pheaſant of* Penſylvania, *and the* Otis Minor. *By Mr.* George Edwards.

To Mr. Peter Collinſon, *F. R. S.*

College of Phyſicians, Jan. 10, 1754.

SIR,

Read Jan. 17, 1754.

ACcording to your requeſt, and by your aſſiſtance, I have drawn up a brief account of the fowl, called a pheaſant in Penſylvania, in order to lay it, together with the birds, before the *Royal Society*.

The coloured print, *Plate* XV. repreſents what is called the pheaſant in Penſylvania and other provinces of North America, tho' it rather belongs to that genus of birds, which in England we call heathcocks, moorgame, or grous. It is near as big as a pheaſant, of a browniſh colour on the head and upper ſide, and white on the breaſt and belly; beautifully variegated with lighter and darker colours on the back, and ſpots of black on the under ſide. Its legs are feathered down to the feet, which will appear by the bird preſerved dry, here preſent, as well as by the print in miniature. As this bird is, in my judgment, wholly unknown to the curious of our country, I